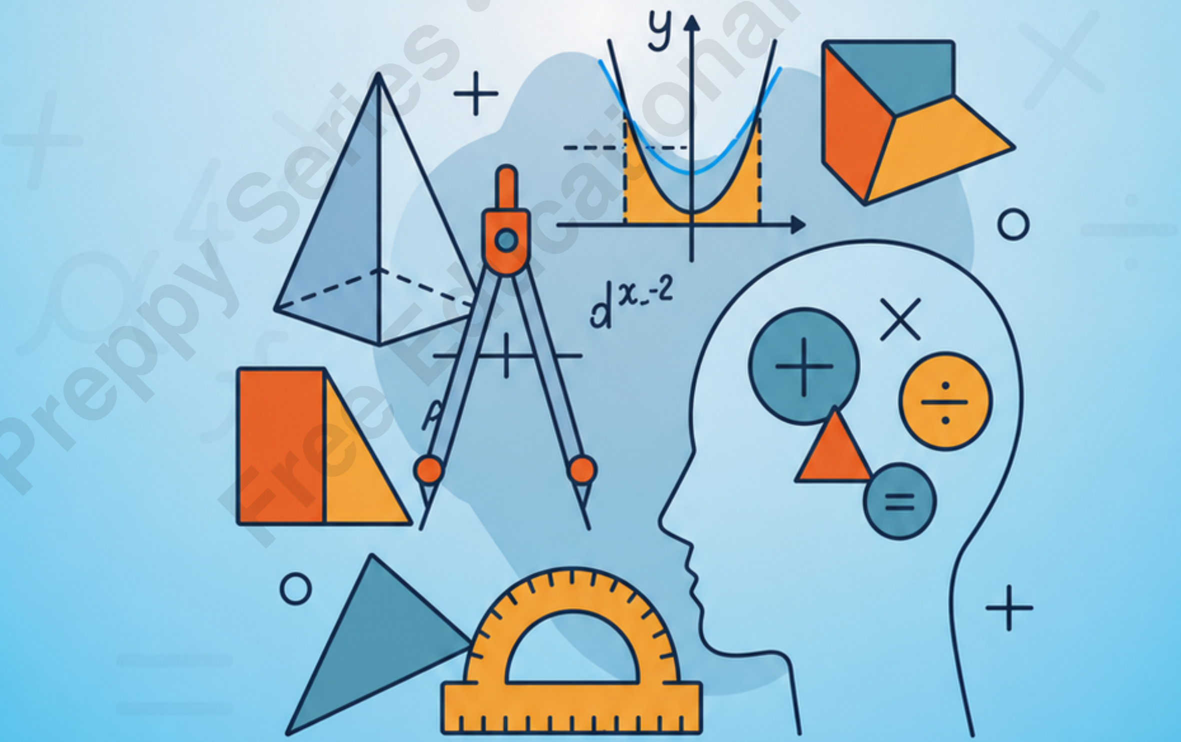
Preppy Series

MATHEMATICS

for Junior High Schools

BECE Past Questions & Solutions

Arranged by Year



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Mathematics

for Junior High Schools

BECE Past Questions & Solutions

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Compiled by

preppyonline.org

2025 Edition

Preface

This book is designed to be a helpful resource for junior high school students, especially those preparing for the Basic Education Certificate Examination (BECE) of the West African Examinations Council (WAEC).

It features

- a collection of complete BECE Mathematics past papers—both objective and essay-type questions—arranged by year
- detailed and thoughtfully explained solutions
- short notes on important facts and methods frequently used on the exams

Our philosophy is simple: learning Mathematics should go beyond memorizing procedures or blindly applying algorithms and tricks. True understanding comes from engaging with problems, exploring different approaches, and appreciating how and why solutions work. For this reason, each solution is presented

- in a step by step fashion
- with clear reasoning
- with multiple perspectives where appropriate
- and with an emphasis on building strong conceptual foundations

Special attention has been given to clarity, with well-drawn diagrams and structured workings that carry the student along through the solution process.

In line with the goal of widening access to quality educational materials, this resource is available at preppyonline.org as a free PDF download. It is our hope that students, teachers, and independent learners alike will find it useful not just for exam preparation, but for developing confidence and deeper insight into Mathematics.

Comments, suggestions, and other enquiries may be sent to preppyonline@gmail.com

Contents

Chapter 1

2025 Paper 1

1. What is the missing number in the sequence: $-5, -2, 1, \ldots, 7$?

A. 2

B. 3

C. 4

D. 5

2. The population of a town is 56782. What is this number to **three** significant figures?

A. 567

B. 568

C. 56700

D. 56800

3. When 0.24 is expressed in the lowest form as $\frac{a}{b}$, the denominator is

A. 2

B. 5

C. 25

D. 125

4. A farmer feeds 20 goats with 500 kg of cassava. How many goats can be fed with 200 kg of cassava?

A. 2

B. 5

C. 8

D. 10

5. A man spends GH¢ 560.00 out of his weekly wage of GH¢ 700.00 and saves the rest. What percentage did he save?

A. 10%

B. 15%

C. 20%

D. 25%

6. Two interior angles of a triangle are $(3x-10)^\circ$ and $(4x+20)^\circ$. Find an expression for the third angle.

A. $(170 - 7x)^\circ$

B. $(150 - 5x)^\circ$

C. $(120 - 7x)^\circ$

D. $(100 - 5x)^\circ$

7. Ama has 4 one cedi notes and orders an ice cream for GH¢ 1.75 and two toffees at 50 GP **each**. How much does she have left?

A. GH¢ 1.25

B. GH¢ 1.75

C. GH¢ 2.25

D. GH¢ 2.75

8. If $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$, find $2\mathbf{a} - \mathbf{b}$.

A. $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

B. $\begin{pmatrix} -9 \\ -1 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

D. $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

9. Find the rule for the mapping:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & t \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ 9 & 20 & 31 & 42 & \cdots & - \end{array}$$

A. $t \longrightarrow 10t - 1$

B. $t \longrightarrow 8t + 1$

C. $t \longrightarrow 11t - 2$

D. $t \longrightarrow 7t + 2$

10. In an examination, Abu answered **nine** questions in 2 hours. He spent 20 minutes on the first question and the same time on **each** of the remaining questions. How many minutes did he spend on **each** of the **other** questions?

A. 8.0 minutes
B. 10.0 minutes
C. 12.0 minutes
D. 12.5 minutes

11. A boy walked round a circular pond once. If the radius of the pond is 28 m, find the distance covered. [Take $\pi = \frac{22}{7}$]

A. 44 m
B. 88 m
C. 176 m
D. 252 m

12. Solve $2^x = 8 \times 2^0$.

A. $x = 3$
B. $x = 2$
C. $x = -2$
D. $x = -3$

13. A survey shows that 28% of all the men in a village are vegetarian. What is the probability that a man selected at random from the village is a vegetarian?

A. $\frac{7}{25}$
B. $\frac{41}{50}$
C. $\frac{1}{2}$
D. 1

14. A trader received a commission of 5% on goods sold at GH¢ 25,000.00. Find the commission.

A. GH¢ 1,250.00
B. GH¢ 1,200.00
C. GH¢ 1,100.00
D. GH¢ 1,000.00

15.

Which of the following inequalities is represented on the number line?

A. $x < 2$
B. $x \leqslant 3$
C. $x > 3$
D. $x \geqslant 2$

16. It costs a carpenter GH¢ 25.00 to make a chair. How much should it be sold to make a profit of 40%?

A. GH¢ 15.00
B. GH¢ 35.00
C. GH¢ 40.00
D. GH¢ 50.00

17. The area of a rectangular card is 15 cm^2. If **each** side of the card is enlarged by a scale factor of 3, find the area of the enlarged card.

A. 45 cm^2
B. 75 cm^2
C. 90 cm^2
D. 135 cm^2

18. Ama is three times as old as Kofi. The sum of their ages is 40. How old is Ama?

A. 10 years
B. 30 years
C. 37 years
D. 43 years

19. Expand and simplify: $2(3a + 1) - 3(4a - 3)$.

A. $11 - 5a$
B. $11 - 6a$
C. $11 + 5a$
D. $11 + 6a$

20. Given that $0.03 \times y = 2.4$, find the value of y.

A. 0.08
B. 0.8
C. 8.0
D. 80.0

21. Charles and Helen started a business with an amount of GH¢ 7,000.00. If their contributions were in the ratio 4:3 respectively, find Helen's contribution.

A. GH¢ 2,500.00
B. GH¢ 3,000.00
C. GH¢ 4,000.00
D. GH¢ 5,000.00

22. A box contains 10 green and 8 white balls of the same size. If a ball is selected at random from the box, what is the probability that it is green?

A. $\frac{1}{10}$
B. $\frac{4}{5}$
C. $\frac{5}{9}$

D. $\frac{4}{9}$

23. Solve: $4x - 2(x + 5) = -10$.

A. $x = -10$

B. $x = 0$

C. $x = \frac{1}{2}$

D. $x = 2$

24. A trader sold half of a piece of cloth and used two-fifths of the remaining to sew a dress. What fraction of the cloth was left?

A. $\frac{1}{10}$

B. $\frac{3}{10}$

C. $\frac{1}{5}$

D. $\frac{1}{10}$

25. If $2y = 5 - 3x$, find x when $y = 1$.

A. $-2\frac{1}{3}$

B. -1

C. 0

D. 1

26. Esi made sales of 15 twenty cedi notes, 14 ten cedi notes and 15 two cedi notes. Find her total sales.

A. GH¢ 305.00

B. GH¢ 440.00

C. GH¢ 470.00

D. GH¢ 740.00

27. A point $(-2, 3)$ is reflected in the x-axis. Find the image of the point.

A. $(-3, -2)$

B. $(-3, 2)$

C. $(-2, -3)$

D. $(-2, 3)$

28. Solve: $3(x - 5) > 15 - 4(8 - x)$.

A. $x < -32$

B. $x < -2$

C. $x < 2$

D. $x < 32$

29. Express 36 as a product of primes.

A. 2×3

B. $2^2 \times 3^2$

C. $2^2 \times 3^3$

D. $2^3 \times 3^2$

30. The points $M(1, 3)$ and $N(4, 5)$ are in the number plane. Find the vector $\overrightarrow{MN}$.

A. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

B. $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$

C. $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$

D. $\begin{pmatrix} -5 \\ -8 \end{pmatrix}$

31. Anowa scored an average of 53 in Science and Mathematics. If she scored 50 and 60 in English Language and Social Studies respectively, find her mean score in all the four subjects.

A. 57

B. 56

C. 55

D. 54

32. Given that $P = \{4, 8, 12, 16, 20\}$ and $Q = \{2, 4, 6, 8, 10\}$, find the product of the members of $(P \cap Q)$.

A. 12

B. 18

C. 24

D. 32

33. A rectangular container with dimensions 5 m by 3 m by 4 m is **two-thirds** full of water. Find the volume occupied by the water in the container.

A. 50 m^3

B. 40 m^3

C. 30 m^3

D. 20 m^3

34. Find the gradient of the line which passes through the points $(2, 3)$ and $(-4, 5)$.

A. -3

B. $-\frac{1}{3}$

C. $\frac{1}{3}$

D. 3

35. Mr. Adu bought 400 bags of maize for his farm animals. If he used 120 bags to feed the animals, find the percentage of the maize left.

A. 70%

B. 60%

C. 50%

D. 40%

36. Find the **largest** value of these numbers: $-1, 0, -6, -3$.

A. 0

B. -1

C. -3

D. -6

37. How long will the simple interest on GH¢ 550.00 at 12% per annum be GH¢ 132.00?

A. 1 year

B. 2 years

C. 4 years

D. 12 years

38. Factorize: $5ay - by + 15a - 3b$.

A. $(y + 3)(5a - b)$

B. $(y + 5)(3a - b)$

C. $(y - 3)(5a + b)$

D. $(y - 5)(3a + b)$

39. Describe the set of $M = \{2, 3, 5, 7, 11, 13, 17, 19\}$ in words.

A. M = {odd numbers less than 20}

B. M = {factors of 19}

C. M = {prime numbers less than 20}

D. M = {whole numbers less than 20}

40. Multiply $(8s - 7)$ by $(8s + 7)$.

A. $64s^2 + 49$

B. $64s^2 - 49$

C. $16s^2 - 42$

D. $16s^2 + 42$

Chapter 2

Solutions to 2025 Paper 1

Answer key

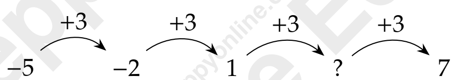
1. C	9. C	17. D	25. D	33. B
2. D	10. D	18. B	26. C	34. B
3. C	11. C	19. B	27. C	35. A
4. C	12. A	20. D	28. C	36. A
5. C	13. A	21. B	29. B	37. B
6. A	14. A	22. C	30. A	38. A
7. A	15. B	23. B	31. D	39. C
8. A	16. B	24. B	32. D	40. B

Solutions

1. Answer: C

We want to find the pattern the numbers in the sequence $-5, -2, 1, \ldots, 7$ follow.

By inspection, we can see that given a term of the sequence, the next term is obtained by adding 3 to the given term.



So, for example, starting with -5, we add 3 to get -2.

Then we add 3 to -2 to get 1.

Adding 3 to 1 gives 4.

And adding 3 to 4 gives 7.

Hence, the missing number is 4.

2. Answer: D

To round 56,782 to three significant figures, we note that the first three significant digits are 5, 6, and 7.

The digit after the 7 is 8, a number greater than 5. Hence, we round up the third digit, 7, to get 8.

So, 567 . . . becomes 568 . . . .

We then replace the remaining digits with zeros to get $56,800$.

3. Answer: C

Writing 0.24 as a fraction gives $\frac{24}{100}$.

To write $\frac{24}{100}$ in its lowest terms, we divide both the numerator and the denominator by their common factor of 4 to get

$$\frac{\cancel{24}^{6}}{\cancel{100}_{25}} = \frac{6}{25}$$

Thus, when 0.24 is expressed in its lowest form as $\frac{a}{b}$, we get $\frac{6}{25}$ and the denominator is 25.

4. Answer: C

The farmer feeds 20 goats with 500 kg of cassava. How many goats can he feed with 200 kg of cassava?

We will solve this in three ways:

- Method 1: Think in terms of equivalent ratios. That is 500 kg : 20 goats as 200 kg : x goats
- Method 2: Find how many goats 1 kg of cassava feeds, then multiply this number by 200 to get the number of goats 200 kg of cassava will feed
- Method 3: Find how many kg of cassava feeds one goat, then divide 200 kg by this number to get how many goats 200 kg of cassava will feed

Method 1

Thinking in terms of equivalent ratios, we have the following problem:

$$500 \text{ kg} : 20 \text{ goats as } 200 \text{ kg} : x \text{ goats}$$

Since each goat eats the same amount of cassava, these ratios are equivalent and we have

$$\frac{500}{20} = \frac{200}{x}$$

Cross-multiplying and solving for x gives

$$\begin{aligned} \frac{500}{20} &= \frac{200}{x} \\ 500x &= 200 \times 20 \\ x &= \frac{200 \times 20}{500} \\ x &= 8 \end{aligned}$$

Method 2

If 500 kg of cassava feeds 20 goats, then 1 kg of cassava will feed

$$\frac{20}{500} \text{ goats}$$

If 1 kg of cassava feeds $\frac{20}{500}$ goats, then 200 kg of cassava will feed

$$\frac{20}{500} \times 200 = 8 \text{ goats}$$

Method 3

If 500 kg of cassava feeds 20 goats, then each goat eats

$$\frac{500}{20} = 25 \text{ kg of cassava}$$

If each goat eats 25 kg of cassava, then how many goats will eat 200 kg of cassava?

$$\frac{200}{25} = 8 \text{ goats}$$

Thus, all three methods give the same answer.

5. Answer: C

The man's weekly wage was GH¢ 700.00.

The amount he spent was GH¢ 560.00.

Therefore, the amount he saved was GH¢ (700 − 560) = GH¢ 140

As a percentage of his wage of GH¢ 700.00, GH¢ 140.00 is

$$\frac{140}{700} \times 100\% = 20\%$$

6. Answer: A

The sum of the three interior angles of a triangle is 180°.

If we know two of the angles, we can obtain the third one by subtracting the other two from the total of 180°.

Doing so gives the third angle as

$$180 - (3x - 10) - (4x + 20)$$

Simplifying the expression gives

$$\begin{aligned} &180 - (3x - 10) - (4x + 20) \\ =&180 - 3x + 10 - 4x - 20 \\ =&180 + 10 - 20 - 3x - 4x \\ =&170 - 7x \end{aligned}$$

7. Answer: A

Ama started with 4 one cedi notes. That means she started with GH¢ 4.00.

This is what she bought:

Items bought	Amount spent
One ice cream at GH¢ 1.75	GH¢ 1.75
Two toffees at 50 GP each	2×50 GP = GH¢ 1.00

Hence, the total amount Ama spent was GH¢ (1.75 + 1.00) = GH¢ 2.75.

Hence, the amount she had left was GH¢ (4.00 − 2.75) = GH¢ 1.25.

8. Answer: A

Since $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$,

$$2\mathbf{a} - \mathbf{b} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

Simplifying the expression using the rules of vector algebra (Section 5.14) gives

$$\begin{aligned} 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \end{pmatrix} &= \begin{pmatrix} 2(-2) \\ 2(1) \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} \end{aligned}$$

9. Answer: C

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & t \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ 9 & 20 & 31 & 42 & \cdots & - \end{array}$$

Method 1

Since this is an objective test, we can try the answer options given to see which of them agrees with the mapping.

Starting with option A, we have $t \to 10t - 1$.

Checking the values of $10t - 1$ for different values of t, we get

t	$10t - 1$	Check
1	$10(1) - 1 = 10 - 1 = 9$	✓
2	$10(2) - 1 = 20 - 1 = 19$	✗

When $t = 2$, $10t - 1 = 19$ which does not agree with what we have in the mapping. Hence, option A is not correct.

Testing option B, we have $t \to 8t + 1$.

Checking the values of $8t + 1$ for different values of t, we get

t	$8t + 1$	Check
1	$8(1) + 1 = 8 + 1 = 9$	✓
2	$8(2) + 1 = 16 + 1 = 17$	✗

When $t = 2$, $8t + 1 = 17$ which does not agree with what we have in the mapping. Hence, option B is not correct.

Testing option C, we have $t \to 11t - 2$.

Checking the values of $11t - 2$ for different values of t, we get

t	$11t - 2$	Check
1	$11(1) - 2 = 11 - 2 = 9$	✓
2	$11(2) - 2 = 22 - 2 = 20$	✓
3	$11(3) - 2 = 33 - 2 = 31$	✓
4	$11(4) - 2 = 44 - 2 = 42$	✓

Option C agrees with all the elements in the mapping.

However, we may try option D just to be extra sure option C is the only option that agrees with all the elements of the mapping.

Testing option D, we have $t \to 7t + 2$.

Checking the values of $7t + 2$ for different values of t, we get

t	$7t + 2$	Check
1	$7(1) + 2 = 7 + 2 = 9$	✓
2	$7(2) + 2 = 14 + 2 = 16$	✗

When $t = 2$, $7t + 2 = 16$ which does not agree with what we have in the mapping. Hence, option D is not correct.

Option C is the only one that agrees with the given mapping.

Method 2

By assessing the given mapping, we observe that the difference between consecutive elements in the domain and co-domain are constant.

That is, in the domain, we have

$$1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4$$

where each successive term increases by 1.

At the same time, in the co-domain, we have

$$9 \xrightarrow{+11} 20 \xrightarrow{+11} 31 \xrightarrow{+11} 42$$

where each successive term increases by 11.

Thus, the given mapping is a linear mapping and its rule is of the form

$$t \to at + b$$

where a and b are constants.

When $t = 1$, we have

$$\begin{aligned} a(1) + b &= 9 \\ a + b &= 9 \end{aligned}$$

When $t = 2$, we have

$$\begin{aligned} a(2) + b &= 20 \\ 2a + b &= 20 \end{aligned}$$

Thus, we have two equations we can solve simultaneously to get a and b:

$$a + b = 9 \qquad (1)$$
$$2a + b = 20 \qquad (2)$$

From equation 1, $b = 9 - a$. Substituting this into equation 2 gives

$$\begin{aligned} 2a + (9 - a) &= 20 \\ 2a + 9 - a &= 20 \\ 2a - a &= 20 - 9 \\ a &= 11 \end{aligned}$$

If $a = 11$, then, from equation 1, we have

$$\begin{aligned} 11 + b &= 9 \\ b &= 9 - 11 = -2 \end{aligned}$$

Thus, the rule of the mapping is $t \to 11t - 2$.

10. Answer: D

We are told that Abu answered 9 questions on the examination and that he spent 2 hours doing so.

We are also told that he spent 20 minutes on the first question and the same amount of time on the remaining questions.

Since 1 hour = 60 minutes, the 2 hours Abu spent on the examination is equivalent to $2 \times 60 = 120$ minutes.

If he spent 20 minutes on the first question, the number of minutes he spent on the remaining 8 questions was $120 - 20 = 100$.

Since he spent the same amount of time on each of the remaining 8 questions, the number of minutes he spent on each one was

$$\frac{100}{8} = 12.5$$

11. Answer: C

The distance the boy walked is equal to the circumference of the circle.

This is given by the formula $2\pi r$, where r is the radius of the circle.

Substituting $r = 28$ m and $\pi = \frac{22}{7}$ into the formula gives

$$\begin{aligned} \text{distance walked} &= 2\pi r \\ &= 2 \times \frac{22}{7} \times 28 \text{ m} \\ &= 176 \text{ m} \end{aligned}$$

12. Answer: A

Since $2^0 = 1$, we have

$$\begin{aligned} 2^x &= 8 \times 2^0 \\ &= 8 \times 1 \\ &= 8 \end{aligned}$$

Since the left-hand side is a power of 2, we would like to write the right-hand side also as a power of 2.

Using the fact that $2^3 = 8$, we have

$$2^x = 8 = 2^3$$

Comparing exponents, we see that $x = 3$.

13. Answer: A

If 28% of all the men in the village are vegetarian, the probability that a man selected at random from the village is a vegetarian is

$$28\% = \frac{28}{100} = \frac{\cancel{28}^{7}}{\cancel{100}_{25}} = \frac{7}{25}$$

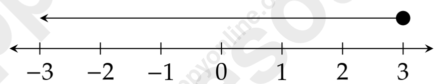
14. Answer: A

The commission the trader received is 5% of the price at which he sold the goods.

That is, 5% of GH¢ 25,000.00. This is the same as

$$\frac{5}{100} \times \text{GH¢ } 25,000.00 = \text{GH¢ } 1,250.00$$

15. Answer: B



The filled circle or black dot on top of 3 means that 3 is part of the set represented by the number line.

The arrow pointing to the left from the black dot also shows that it is numbers to the left of 3 or numbers less than 3 that are part of the set.

Hence, the number line represents the set with 3 and numbers less than 3.

This is the same as the set $\{x : x \leqslant 3\}$.

16. Answer: B

The cost price for the chair is GH¢ 25.00.

To sell it at a profit of 40%, the selling price must be greater than the cost price by 40%.

That means the selling price must be 140% of the cost price.

140% of GH¢ 25.00 is

$$\frac{140}{100} \times \text{GH¢ } 25.00 = \text{GH¢ } 35.00$$

Hence, the selling price must be GH¢ 35.00 for the carpenter to make a profit of 40%.

17. Answer: D

The area, A, of a rectangle is given by $A = lw$, where l is the length of the rectangle and w is the width.

If each side of the rectangle is enlarged by a scale factor of 3, the new length will be $3l$ and the new width will be $3w$.

Hence, the new area after the enlargement will be $3l \times 3w = 9lw$.

Since the original area was lw, the new area of $9lw$ is 9 times the original area.

That means that after enlarging each side of the rectangle by a scale factor of 3, the area will increase by a factor of 9.

Hence, if the original area was 15 cm^2, the new area, after the enlargement, will be

$$15\ \text{cm}^2 \times 9 = 135\ \text{cm}^2$$

18. Answer: B

Given some relationships between Kofi's and Ama's ages, we are asked to find Ama's age. We shall look at two ways of solving this problem.

In the first one, we represent Kofi's age by a variable and use the information given to write an expression for Ama's age. We then use more of the given information to solve for their respective ages.

In the second method, we think in terms of ratios. Since Ama is three times as old as Kofi, the ratio of Ama's age to Kofi's age is $3:1$. We then use this and the other information given to find Ama's age.

Method 1

Let Kofi's age be k.

Then, since Ama is three times as old as Kofi, Ama's age is $3k$.

Since the sum of their ages is 40, we have

$$k + 3k = 40$$

Solving this equation gives

$$\begin{aligned} k + 3k &= 40 \\ 4k &= 40 \\ k &= \frac{40}{4} = 10 \end{aligned}$$

Thus, Kofi is 10 years old and Ama is

$$3 \times 10 = 30 \text{ years old}$$

Method 2

Since Ama is three times as old as Kofi, the ratio of Ama's age to Kofi's age is

$$3:1$$

What fraction of the sum of their ages is Ama's age? That fraction must be

$$\frac{3}{3+1} = \frac{3}{4}$$

Since the sum of their ages is 40 years, Ama's age is

$$\frac{3}{4} \times 40 = 30 \text{ years}$$

19. Answer: B

$$\begin{aligned} 2(3a+1) - 3(4a-3) &= 6a + 2 - 12a + 9 \\ &= 6a - 12a + 2 + 9 \\ &= -6a + 11 \end{aligned}$$

Looking at the answer options given, they are all of the form "11 + something" or "11 − something" so we also want to write our answer in that form.

Since when adding numbers we can add then in any order (Section 5.2.1), we have

$$-6a + 11 = 11 + (-6a) = 11 - 6a$$

20. Answer: D

To solve for y in $0.03 \times y = 2.4$, we can divide both sides by 0.03 to get

$$y = \frac{2.4}{0.03}$$

How do we evaluate $\frac{2.4}{0.03}$?

Noticing that the denominator is smaller than the numerator and that we can turn the denominator, 0.03, into a whole number by multiplying it by 100, we may get rid of the decimals in the fraction by multiplying both the numerator and denominator by 100. Doing so gives

$$y = \frac{2.4}{0.03} = \frac{2.4 \times 100}{0.03 \times 100} = \frac{240}{3} = 80$$

21. Answer: B

Charles and Helen started a business with an amount of GH¢ 7,000.00. If their contributions were in the ratio 4:3 respectively, find Helen's contribution.

If Charles's and Helen's contributions were in the ratio 4:3 respectively, Helen's share of the total contribution must have been

$$\frac{3}{4+3} = \frac{3}{7}$$

Hence, Helen's contribution was

$$\frac{3}{7} \times \text{GH¢ } 7000 = \text{GH¢ } 3000$$

22. Answer: C

The probability that a ball chosen at random from the box is green is

$$\frac{\text{the number of green balls in the box}}{\text{the total number of balls in the box}}$$

Since there are 10 green balls and 8 white balls in the box, the probability that a randomly chosen ball is green is

$$\frac{10}{10+8} = \frac{10}{18} = \frac{5}{9}$$

23. Answer: B

$$\begin{aligned} 4x - 2(x+5) &= -10 \\ 4x - 2x - 10 &= -10 \\ 2x &= -10 + 10 \\ 2x &= 0 \\ x &= \frac{0}{2} = 0 \end{aligned}$$

24. Answer: B

After selling $\frac{1}{2}$ of the piece of cloth, the trader had $\frac{1}{2}$ left.

Of this remaining $\frac{1}{2}$, she used $\frac{2}{5}$ to sew a dress. That means $1 - \frac{2}{5} = \frac{3}{5}$ of the remaining $\frac{1}{2}$ was left.

This is equivalent to

$$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

Thus, $\frac{3}{10}$ of the original piece of cloth was left.

25. Answer: D

Substituting $y = 1$ into $2y = 5 - 3x$ gives

$$\begin{aligned} 2(1) &= 5 - 3x \\ 2 &= 5 - 3x \\ 3x &= 5 - 2 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

26. Answer: C

Esi made sales of 15 twenty cedi notes, 14 ten cedi notes and 15 two cedi notes.

We may record the sales Esi made as follows

Note	Number	Amount
GH¢ 20	15	15 × GH¢ 20 = GH¢ 300
GH¢ 10	14	14 × GH¢ 10 = GH¢ 140
GH¢ 2	15	15 × GH¢ 2 = GH¢ 30

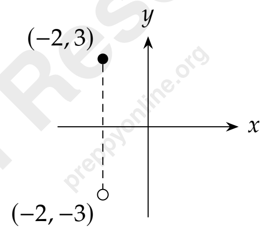
From the table, we can see that the total amount of sales Esi made was

$$\text{GH¢ } (300 + 140 + 30) = \text{GH¢ } 470.00$$

27. Answer: C

We draw a sketch in which the point $(-2, 3)$ is shown with a shaded dot and its reflection is shown with an empty dot.

Thinking of the x-axis as our mirror line, it is clear that the image of the point $(-2, 3)$ should lie below the x-axis as $(-2, 3)$ lies above the x-axis.



From the sketch, the x-coordinate of the image remains the same as that of the original point but its y-coordinate is negated.

28. Answer: C

To solve the inequality, we expand the brackets, group like terms, and simplify.

$$\begin{aligned} 3(x-5) &> 15 - 4(8-x) \\ 3x - 15 &> 15 - 32 + 4x \\ 3x - 4x &> 15 - 32 + 15 \\ -x &> -2 \end{aligned}$$

To isolate x in the expression, we can multiply or divide both sides of the inequality by -1. However, we must remember that multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign. Hence,

$$-x > -2 \quad \text{means that} \quad x < 2$$

29. Answer: B

To write 36 as a product of primes, we may first write it as a product of any factors and then write those factors also as a product of factors until we are only left with prime factors.

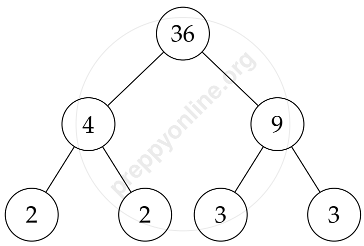
For example,

$$36 = 4 \times 9$$

However, since $4 = 2 \times 2$ and $9 = 3 \times 3$, we may express 36 as

$$\begin{aligned} 36 &= 4 \times 9 \\ &= 2 \times 2 \times 3 \times 3 \\ &= 2^2 \times 3^2 \end{aligned}$$

This calculation is the same as what we get from the factor tree (Section 5.2.2) for 36 below:



30. Answer: A

The position vectors corresponding to the points $M(1,3)$ and $N(4,5)$ are

$$\overrightarrow{OM} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{ON} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

By the rules of vector addition,

$$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$$

Since $\overrightarrow{MO} = -\overrightarrow{OM}$, we have

$$\begin{aligned} \overrightarrow{MN} &= -\overrightarrow{OM} + \overrightarrow{ON} \\ &= -\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -1+4 \\ -3+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{aligned}$$

31. Answer: D

To find Anowa's mean score in all the four subjects, we may first find her total score in the four subjects, and divide the result by 4.

Since her average score in Science and Mathematics was 53, her total score in those two subjects was $2 \times 53 = 106$.

Adding her score in English Language and Social Studies gives the total score in all the four subjects as

$$106 + 50 + 60 = 216$$

Hence, her mean score in the four subjects was

$$\frac{216}{4} = 54$$

32. Answer: D

If $P = \{4, 8, 12, 16, 20\}$ and $Q = \{2, 4, 6, 8, 10\}$, the set that contains members that are in both P and Q, is

$$P \cap Q = \{4, 8\}$$

The product of the members of $P \cap Q$ is what we get when we multiply all the members of $P \cap Q$ together:

$$4 \times 8 = 32$$

33. Answer: B

The volume of the rectangular container is

$$5 \text{ m} \times 3 \text{ m} \times 4 \text{ m}$$

$\frac{2}{3}$ of this volume is

$$\frac{2}{3} \times 5 \text{ m} \times 3 \text{ m} \times 4 \text{ m} = 40 \text{ m}^3$$

Hence, the volume occupied by the water is 40 m^3.

34. Answer: B

The gradient (Section 5.12.1) of a line that passes through two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2}$$

So, when we are given the coordinates of two points, we can take one of them as (x_1, y_1) and the other as (x_2, y_2) and apply the formula above to find the gradient.

Taking $(2, 3)$ as (x_1, y_1) and $(-4, 5)$ as (x_2, y_2), we find the gradient as

$$\frac{5-3}{-4-2} = \frac{2}{-6} = -\frac{1}{3}$$

35. Answer: A

Mr. Adu bought 400 bags of maize.

If he used 120 bags to feed his animals, the number of bags remaining must have been $400 - 120 = 280$.

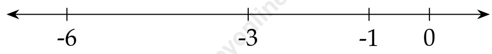
Thus, the percentage of maize left was

$$\frac{280}{400} \times 100\% = 70\%$$

36. Answer: A

The largest number among $-1, 0, -6, -3$ is 0.

If we plot the numbers on a number line, we will find all the other numbers to the left of 0. That is a nice way to see that 0 is the largest among them.



37. Answer: B

How long will the simple interest on GH¢ 550.00 at 12% per annum be GH¢ 132.00?

The formula for simple interest is

$$\text{simple interest} = \text{principal} \times \text{rate} \times \text{time}$$

We are given a principal of GH¢ 550.00,

a rate of 12% per annum,

and a simple interest of GH¢ 132.00.

Substituting these into the formula gives

$$\text{GH¢ } 132.00 = \text{GH¢ } 550.00 \times \frac{12}{100} \times \text{time}$$

Solving for time gives

$$\text{time} = \frac{\text{GH¢ } 132.00 \times 100}{\text{GH¢ } 550.00 \times 12} = 2$$

Thus, it will take 2 years to get a simple interest of GH¢ 132.00 on a principal of GH¢ 550.00 at a rate of 12% per annum.

38. Answer: A

$$\begin{aligned} 5ay - by + 15a - 3b &= (5ay - by) + (15a - 3b) \\ &= y(5a - b) + 3(5a - b) \\ &= (5a - b)(y + 3) \end{aligned}$$

39. Answer: C

How would we describe the set $M = \{2, 3, 5, 7, 11, 13, 17, 19\}$ in words?

It is the set of prime numbers less than 20.

Since this is an objective test in which we are given several answer options, we may walk through the options to see which of them best describes the set.

Option A: $M = \{\text{odd numbers less than 20}\}$

This cannot be correct because the set M contains 2, which is not an odd number.

Option B: $M = \{\text{factors of 19}\}$

This cannot be correct because M contains 2, which is not a factor of 19.

M also contains other numbers that are not factors of 19 but finding only one number that is not a factor is enough to disqualify this answer option.

Option D: $M = \{\text{whole numbers less than 20}\}$

This cannot be correct because there are whole numbers less than 20 that are not in the set M. Some examples are 1, 4, and 6.

Option C is the only one that describes the given set.

40. Answer: B

Method 1

If we notice that $(8s-7)(8s+7)$ is the factorization of a difference of two squares, we can immediately recognize that it is equal to

$$(8s)^2 - 7^2 = 64s^2 - 49$$

Method 2

We may expand $(8s-7)(8s+7)$ using the distributive law to get

$$\begin{aligned} (8s - 7)(8s + 7) &= 8s(8s + 7) - 7(8s + 7) \\ &= 8s(8s) + 7(8s) - 7(8s) - 7(7) \\ &= 64s^2 - 49 \end{aligned}$$

Chapter 3

2025 Paper 2

1. (a) Given that P = {multiples of 3} and Q = {positive even numbers} are subsets of $\mu = \{x : 1 < x \leq 20$, where x is a counting number$\}$:

i. list the elements in $P \cap Q$;

ii. list all the subsets in $P \cap Q$.

(b) If $\frac{1}{y} = 3k - \frac{2}{x}$,

i. make y the subject of the relation.

ii. using the result in 1(b)i, find the value of y when $x = -1$ and $k = 2$.

2. (a) Evaluate $\frac{4000 \times 0.35}{0.05}$, leaving the answer in standard form.

(b) Mr. Boakye gets 10% commission on type P house he sells and 15% on type Q house. He sells 3 type P houses at GH¢ 700,000.00 each and 4 type Q at GH¢ 1,400,000.00 each. Calculate the total commissions he makes.

3. (a) Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} x \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, find:

i. the value of x, if $2\mathbf{a} + \mathbf{b} = \mathbf{c}$;

ii. $\mathbf{d} = \mathbf{c} - 3\mathbf{a}$;

iii. $|\mathbf{d}|$.

(b) A Polytank contains 4500 litres of water and $\frac{1}{5}$ of the water is used for cleaning.

i. Find the volume of water used for cleaning.

ii. What percentage of water is left in the tank?

4. (a) A woman borrowed GH¢ 5,300.00 to pay for her child's university fees. If she borrowed at a rate of 8% simple interest per annum for 9 months, find the interest paid.

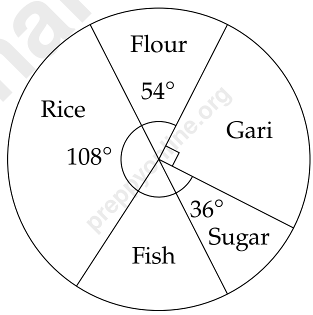
(b) A father shared his piece of land to his **three** children. The first child had $\frac{2}{5}$ of the land and the second had 5 acres more than the first. If the third child had 20 acres, find how many acres of land the

i. father shared;

ii. first child received;

iii. second child received.

5. (a) The pie chart shows the weight (in kg) of items Mrs. Mensah bought for her household.



NOT DRAWN TO SCALE

i. What angle represents fish?

ii. If she bought a total of 20 kg of items,

(α) what is the weight of flour bought?

(β) express, correct to **one** decimal place, the weight of sugar as a perecentage of the weight of rice.

(b) In a class of 30 students, five wear glasses. If a student is selected at random from the class, what is the probability that the student does not wear glasses?

6. Adamu was travelling a distance of 40 km from Kadungu to Datanu. Sixty minutes after starting the journey, he made a stop at Cooltown, 10 km from Kadungu to rest for 30 minutes. He then continued the journey from Cooltown and reached Datanu 60 minutes later.

(a) Using a scale of 2 cm to 20 minutes on the horizontal axis and 2 cm to 5 km on the vertical axis, draw a distance-time graph for Adamu's journey.

(b) Use the graph to determine the:

i. distance from Cooltown to Datanu;

ii. total time time (in minutes), taken by Adamu to make the whole journey, including the rest time

iii. average speed of Adamu from Cooltown to Datanu.

(c) If Adamu did **not** rest but travelled to Datanu within the time, what was his average speed?

Chapter 4

Solutions to 2025 Paper 2

Question 1

(a) Since $\mu = \{x : 1 < x \leqslant 20,$ where x is a counting number$\}$,

$$\mu = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

Since $P = \{$multiples of 3$\}$ and it is a subset of μ,

$$P = \{3, 6, 9, 12, 15, 18\}$$

Since $Q = \{$positive even numbers$\}$ and it is a subset of μ,

$$Q = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

(i) $P \cap Q$ is the set of elements that are in both P and Q. Hence,

$$P \cap Q = \{6, 12, 18\}$$

(ii) Since $P \cap Q = \{6, 12, 18\}$, its subsets are the following:

$$\{\}, \{6\}, \{12\}, \{18\}, \{6, 12\}, \{6, 18\}, \{12, 18\}, \{6, 12, 18\}$$

Recalling that a set with n elements has 2^n subsets (Section 5.1.4), we should expect the set $P \cap Q = \{6, 12, 18\}$ to have $2^3 = 8$ subsets. Counting the number of subsets listed above, we get 8, as expected.

(b) (i) To make y the subject of the relation $\frac{1}{y} = 3k - \frac{2}{x}$, we begin by writing the right-hand side as a single fraction.

Doing so gives

$$\frac{1}{y} = 3k - \frac{2}{x}$$

$$\frac{1}{y} = \frac{3kx - 2}{x}$$

Cross-multiplying and simplifying, we get

$$x = y(3kx - 2)$$

$$y = \frac{x}{3kx - 2}$$

(ii) using the result above, when $x = -1$ and $k = 2$,

$$y = \frac{-1}{3(2)(-1) - 2} = \frac{-1}{-6 - 2} = \frac{-1}{-8} = \frac{1}{8}$$

Question 2

(a)

$$\frac{4000 \times 0.35}{0.05} = \frac{(4 \times 10^3) \times (35 \times 10^{-2})}{5 \times 10^{-2}}$$

$$= \frac{(4 \times 10^3) \times (\cancel{35}^{7} \times \cancel{10^{-2}}^{1})}{\cancel{5}_1 \times \cancel{10^{-2}}_1}$$

$$= 28 \times 10^3$$

Writing 28×10^3 in standard form gives 2.8×10^4.

(b) Since Mr. Boakye sold 3 type P houses at GH¢ 700,000.00 each, the total amount of sales from type P houses was

$$\text{GH¢ } 700,000.00 \times 3$$

If his commission on each of those houses was 10%, his total commission on type P houses was

$$\text{GH¢ } 700,000 \times 3 \times \frac{10}{100} = \text{GH¢ } 700,0\cancel{00} \times 3 \times \frac{10}{1\cancel{00}}$$

$$= \text{GH¢ } 7,000 \times 3 \times 10$$

$$= \text{GH¢ } 210,000.00$$

Since Mr. Boakye sold 4 type Q houses at GH¢ 1,400,000.00 each, the total amount of sales from type Q houses was

$$\text{GH¢ } 1,400,000.00 \times 4$$

If his commission on each of those houses was 15%, his total commission on type Q houses was

$$\begin{aligned} 1,400,000 \times 4 \times \frac{15}{100} &= 1,400,0\cancel{00} \times 4 \times \frac{15}{1\cancel{00}} \\ &= 14,000 \times 4 \times 15 \\ &= \text{GH¢ } 840,000.00 \end{aligned}$$

His total commission on all the houses he sold is given by

total commission on type P houses +
total commission on type Q houses

That is,

$$\text{GH¢ } 210,000 + \text{GH¢ } 840,000 = \text{GH¢ } 1,050,000.00$$

Question 3

(a) We are given $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} x \\ -3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

(i) If $2\mathbf{a} + \mathbf{b} = \mathbf{c}$, then

$$\begin{aligned} 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} x \\ -3 \end{pmatrix} &= \begin{pmatrix} 7 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} x \\ -3 \end{pmatrix} &= \begin{pmatrix} 7 \\ 3 \end{pmatrix} \\ \begin{pmatrix} x \\ -3 \end{pmatrix} &= \begin{pmatrix} 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\ \begin{pmatrix} x \\ -3 \end{pmatrix} &= \begin{pmatrix} 3 \\ -3 \end{pmatrix} \end{aligned}$$

Equating the x components of the vectors on the left-hand side and the right-hand side, we get $x = 3$.

(ii) If $\mathbf{d} = \mathbf{c} - 3\mathbf{a}$, then

$$\begin{aligned} \mathbf{d} &= \begin{pmatrix} 7 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -6 \end{pmatrix} \end{aligned}$$

(iii) Using the formula for the length of a vector (Section 5.14.2),

$$|\mathbf{d}| = \left|\begin{pmatrix} 1 \\ -6 \end{pmatrix}\right| = \sqrt{1^2 + (-6)^2} = \sqrt{1 + 36} = \sqrt{37}$$

(b) (i) The volume of water used for cleaning is $\frac{1}{5}$ of 4500 litres. This evaluates to

$$\frac{1}{5} \times 4500 \text{ litres} = 900 \text{ litres}$$

(ii) After using $\frac{1}{5}$ of the water for cleaning, the fraction left was

$$1 - \frac{1}{5} = \frac{4}{5}$$

Writing this as a percentage gives

$$\frac{4}{5} \times 100\% = \frac{4}{\cancel{5}_1} \times \cancel{100\%}^{20} = 80\%$$

Question 4

(a) To find the interest the woman paid, we may use the formula for simple interest (Section 5.4):

$$\text{simple interest} = \text{principal} \times \text{rate} \times \text{time}$$

We are told that the principal or the amount she borrowed was GH¢ 5,300.00.

The rate was 8% per annum.

The time was 9 months.

Since the rate was given per annum or per year, we have to convert the time into years. Since there are 12 months in a year, the 9 months given is equivalent to $\frac{9}{12}$ years.

Putting these values into the formula, we get

$$\begin{aligned} \text{simple interest} &= \text{GH¢ } 5,300 \times 8\% \times \frac{9}{12} \\ &= \text{GH¢ } 5,300 \times \frac{8}{100} \times \frac{9}{12} \\ &= \text{GH¢ } 5,3\cancel{00} \times \frac{\cancel{8}^2}{1\cancel{00}} \times \frac{9}{\cancel{12}_3} \\ &= \text{GH¢ } \frac{53 \times 2 \times 9}{3} \\ &= \text{GH¢ } 318.00 \end{aligned}$$

(b) Let the number of acres of land the father shared be a.

Since the first child had $\frac{2}{5}$ of the land, the area of land he got was

$$\frac{2}{5}a$$

If the second child got 5 more acres than the first, then he had

$$\frac{2}{5}a + 5$$

The third child got 20 acres.

(i) Adding the number of acres of land each child received should give us the number of acres of land the father shared. That is,

$$a = \frac{2}{5}a + \frac{2}{5}a + 5 + 20$$

Solving the equation gives

$$\begin{aligned} a &= \frac{2}{5}a + \frac{2}{5}a + 5 + 20 \\ a &= \frac{4}{5}a + 25 \\ a - \frac{4}{5}a &= 25 \\ \frac{1}{5}a &= 25 \\ a &= 25 \times 5 \\ a &= 125 \end{aligned}$$

Thus the father shared a land of 125 acres.

(ii) Since the first child received $\frac{2}{5}$ of the land the father shared, his share was

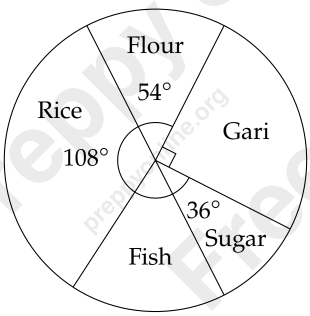
$$\begin{aligned} \frac{2}{5} \text{ of } 125 \text{ acres} &= \frac{2}{5} \times 125 \text{ acres} \\ &= 50 \text{ acres} \end{aligned}$$

(iii) Since the second child got 5 acres more than the first child, his share of the land was

$$(50 + 5) \text{ acres} = 55 \text{ acres}$$

Question 5

(a) The pie chart shows the weight (in kg) of items Mrs. Mensah bought for her household.



NOT DRAWN TO SCALE

(i) Since the sum of the sectoral angles in the pie chart must be 360°, we can get the sectoral angle for fish by subtracting the sum of all the other sectoral angles from 360°.

$$\begin{aligned} \text{angle for fish} &= 360^\circ - (108^\circ + 54^\circ + 90^\circ + 36^\circ) \\ &= 360^\circ - 288^\circ \\ &= 72^\circ \end{aligned}$$

(ii) (α) Since the sectoral angle for flour is 54°, the weight of flour she bought is

$$\begin{aligned} \frac{54^\circ}{360^\circ} \times 20 \text{ kg} &= \frac{54^\circ}{\cancel{360^\circ}_{18}} \times \cancel{20}^{1} \text{ kg} \\ &= \frac{54}{18} \text{ kg} \\ &= 3 \text{ kg} \end{aligned}$$

(β)

$$\frac{\text{weight of sugar}}{\text{weight of rice}} = \frac{\text{sectoral angle for sugar}}{\text{sectoral angle for rice}}$$

Since the sectoral angle of sugar is 36° and the sectoral angle for rice is 108°, the weight of sugar as a percentage of the weight of rice is

$$\begin{aligned} \frac{36^\circ}{108^\circ} \times 100\% &= \frac{1}{3} \times 100\% \\ &= 33\frac{1}{3}\% \\ &= 33.3\% \text{ to 1 decimal place} \end{aligned}$$

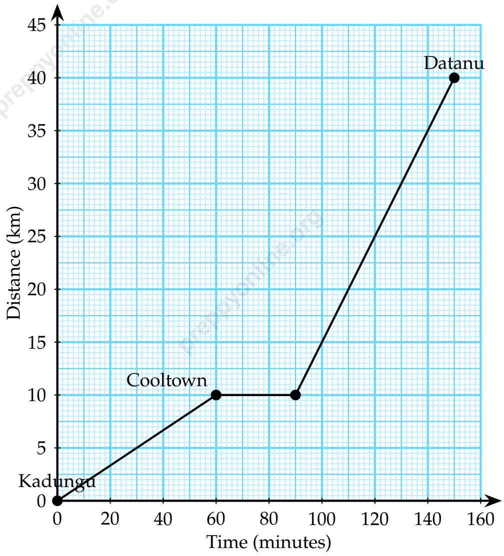
(b) Since only 5 of the 30 students wear glasses, the number of students who don't wear glasses is 30 − 5 = 25.

Hence, the probability that a randomly selected student does not wear glasses is

$$\frac{25}{30} = \frac{5}{6}$$

Question 6

(a) Using a scale of 2 cm to 20 minutes on the horizontal axis and 2 cm to 5 km on the vertical axis, we draw the following distance-time graph for Adamu's journey.



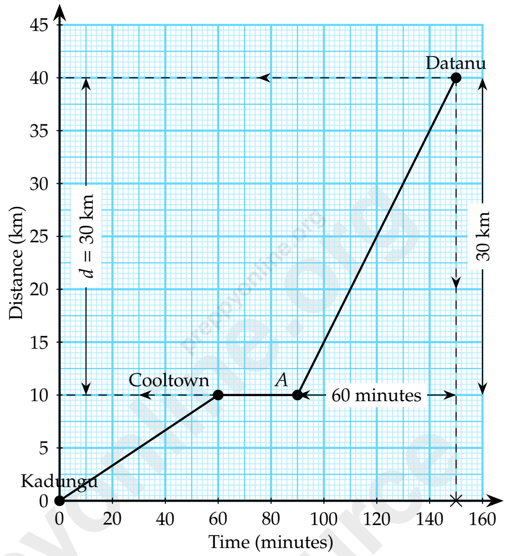
At the time Adamu starts his journey, he is at Kadungu. This is indicated by the dot at the origin, (0, 0), in the bottom left corner of the graph.

We are told that 60 minutes after starting the journey, he made a stop at Cooltown, 10 km from Kadungu. This is indicated by the dot labeled "Cooltown." This dot has coordinates (60, 10).

We are then told that he rested at Cooltown for 30 minutes. This is indicated by the flat line starting from the dot labeled "Cooltown."

After staying in Cooltown for 30 minutes, Adamu sets off for Datanu and arrives after 60 minutes. His final destination, Datanu, is indicated by the dot in the top right corner of the graph.

(b) We draw the graph again with extra lines and labels to show how to use it to answer the given questions.



(i) Using the graph, we can get the distance from Cooltown to Datanu by looking at the vertical distance between where Cooltown is and where Datanu is. We have labelled this distance d on the graph.

Reading the values on the distance axis and subtracting, we can see that this distance is $(40 - 10) = 30$ km.

(ii) At the end of the journey, Adamu was at Datanu. Drawing a vertical line from the point labelled Datanu to the time axis gives us the time he took to make the whole journey.

The point at which the vertical line from Datanu meets the time axis is the 150-minute mark, marked with a cross (×) on the time axis.

So the total time he took was 150 minutes.

(iii) We can get Adamu's average speed from Cooltown to Datanu by taking the gradient of the line that represents that part of his journey.

The part of the graph that represents that part of his journey is the line from the point labelled A to Datanu.

The gradient of that line is

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{30 \text{ km}}{60 \text{ minutes}} = \frac{1}{2} \text{ km per minute}$$

Hence, Adamu's average speed from Cooltown to Datanu was $\frac{1}{2}$ km per minute.

Since 60 minutes is 1 hour, we could also write the gradient as follows:

$$\frac{30 \text{ km}}{60 \text{ minutes}} = \frac{30 \text{ km}}{1 \text{ hour}} = 30 \text{ km per hour}$$

Hence, Adamu's average speed from Cooltown to Datanu was also 30 km per hour.
$\frac{1}{2}$ km per minute equals 30 km per hour.

(c)

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

If Adamu travelled from Kadungu to Datanu, a distance of 40 km, in a time of 150 minutes, his average speed would be

$$\frac{40 \text{ km}}{150 \text{ minutes}} = \frac{4}{15} \text{ km per minute}$$

Since 60 minutes = 1 hour, the 150 minutes Adamu travelled is equivalent to

$$\frac{150}{60} \text{ hours} = \frac{5}{2} \text{ hours}$$

We can use this to calculate his average speed as

$$\frac{40 \text{ km}}{\frac{5}{2} \text{ hour}}$$

Since dividing by a fraction is equivalent to multiplying by its reciprocal, we may simplify this expression as

$$\begin{aligned}\frac{40 \text{ km}}{\frac{5}{2} \text{ hour}} &= 40 \times \frac{2}{5} \text{ km per hour} \\ &= 16 \text{ km per hour}\end{aligned}$$

Hence, Adamu's average speed would also be 16 km per hour.

Chapter 5

Quick Revision Notes

5.1 Sets

A set is a well-defined collection of unique objects.

An example of a set is the set of counting numbers less than 5: $\{1, 2, 3, 4\}$.

Another example is the set of colors in a rainbow: $\{\text{red}, \text{orange}, \text{yellow}, \text{green}, \text{blue}, \text{indigo}, \text{violet}\}$.

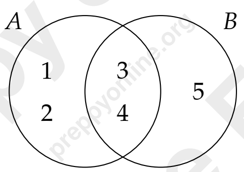
Note that the elements of a set must be unique. That means we can't have repeated elements in a set. For example, $\{1, 2, 2, 3\}$ is not set because 2 is repeated.

Sets are often denoted by capital letters, e.g., $E = \{2, 4, 6, 8\}$.

5.1.1 Venn diagrams

Venn diagrams are a useful tool for showing the relationships between sets.

If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$, we can represent the relationship between A and B with the diagram below.



Notice that because the elements 3 and 4 are in both sets, they are shown in the part of the diagram where the two circles that represent the two sets overlap.

5.1.2 Types of sets

Finite sets

A finite set is a set with a countable number of elements. That means its number of elements can be represented by a non-negative integer such as 0, 1, 2, 3, and so on.

An example of a finite set is the set of vowels in the English alphabet: $\{a, e, i, o, u\}$.

Infinite sets

An infinite set is a set with an unlimited number of elements. That means its number of elements cannot be represented by a natural number.

An example of an infinite set is the set of whole numbers, $\{0, 1, 2, 3, 4, 5, \ldots\}$. This set is infinite because there is no such thing as the largest whole number. Hence, the set is endless.

The empty set

The empty set is the set with no elements. That means its number of elements is 0.

They empty set is denoted by empty curly braces, $\{\}$. It is also denoted by $\varnothing$.

Singletons

A singleton is a set with exactly one element. That means its number of elements is 1.

Examples of singleton sets are the following

$$\{8\} \quad \{\text{A}\} \quad \{0\} \quad \{\text{orange}\}$$

Is the set $\{\{1, 2, 3\}\}$ a singleton?

Yes, it is. It has only one element, the set $\{1, 2, 3\}$.

5.1.3 Subsets

Given a set A, a subset of A is a set whose elements are also elements of A.

For instance, the set $\{3, 4\}$ is a subset of $\{1, 2, 3, 4, 5\}$ because each of its elements—3 and 4—is an element of the set $\{1, 2, 3, 4, 5\}$.

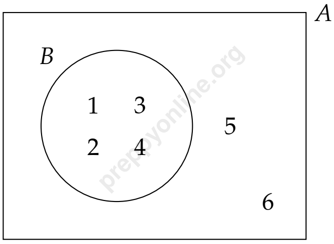
Is $\{1, 2, 3, 4, 5\}$ a subset of $\{1, 2, 3, 4, 5\}$?

Yes, it is. Using the definition above, we have to check if every element of $\{1, 2, 3, 4, 5\}$ is an element of $\{1, 2, 3, 4, 5\}$. As this is obviously the case, $\{1, 2, 3, 4, 5\}$ is a subset of itself.

By this reasoning, every set is a subset of itself.

We write $A \subseteq B$ to denote that A is a subset of B.

The diagram below shows set A, represented by the rectangle, and set B represented by the circle. From the diagram, set B is a subset of set A because every element in B is also in A.



The elements of B are 1, 2, 3, 4, while the elements of A are 1, 2, 3, 4, 5, 6.

Proper subsets

Sometimes we only want subsets of a set that are not the set itself. These are called proper subsets. Hence, though every set is a subset of itself, it is not a proper subset of itself.

$\{3,4,5\}$ is a proper subset of $\{1,2,3,4,5\}$ but $\{1,2,3,4,5\}$ is not a proper subset of $\{1,2,3,4,5\}$.

5.1.4 The number of subsets of a set

The number of subsets of a set with n elements is 2^n.

For example, the set $\{1,2,3\}$ has $2^3 = 8$ subsets because it has 3 elements. The 8 subsets are the following:

$$\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

Notice that the list of subsets includes the empty set, $\{\}$, and the set itself $\{1,2,3\}$. The empty set is a subset of every set. Every set is also a subset of itself.

Since $\{1,2,3,4,5\}$ has 5 elements, it has $2^5 = 32$ subsets. Can you list them?

5.1.5 Comparing sets

Equal sets

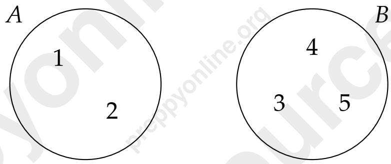
Two sets are equal if they have the same elements. For example, $\{1,2,3\}$ and $\{3,2,1\}$ are equal because they have the same elements. This means the order in which the elements of a set are arranged does not matter. Once two sets have the same elements, they are equal.

Equivalent sets

Two sets are equivalent if they have the same number of elements. For example, $\{1,2,3\}$ and $\{4,5,6\}$ are equivalent because they have the same number of elements, 3.

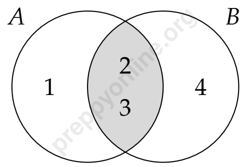
Disjoint sets

Two sets are called disjoint if they have no elements in common. For example, $A = \{1,2\}$ and $B = \{3,4,5\}$ are disjoint because they have no elements in common.



Intersecting sets

If two sets are not disjoint, then they are intersecting sets. Intersecting sets are sets that have elements in common. For example, $A = \{1,2,3\}$ and $B = \{2,3,4\}$ are intersecting sets because they have 2 and 3 in common. Their intersection is shaded in the Venn diagram below.



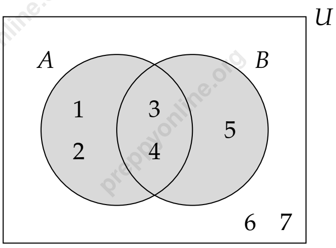
5.1.6 Operations on sets

Union

The union of sets A and B, denoted $A \cup B$, is the set that contains all the elements of both sets.

For example, the union of $A = \{1,2,3\}$ and $B = \{2,3,4\}$ is $A \cup B = \{1,2,3,4\}$.

Let A and B be subsets of the universal set $U = \{1,2,3,4,5,6,7\}$. If $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$ and the relationships between the sets can be represented by the Venn diagram below. The shaded region represents the union.

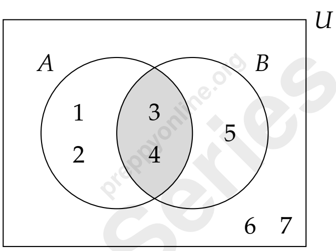


Intersection

The intersection of sets A and B, denoted $A \cap B$, is the set that contains all the elements that are in both sets.

For example, the intersection of $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ is $A \cap B = \{2, 3\}$.

Let A and B be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6, 7\}$. If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3, 4\}$ and the relationships between the sets can be represented by the Venn diagram below. The shaded region represents the intersection.

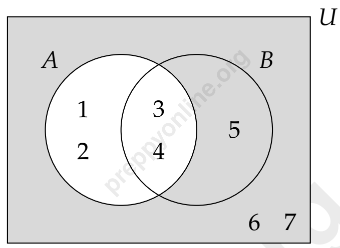


Complement

The complement of a set is the set that contains all the elements that are not in the set. For example, the complement of $A = \{1, 2, 3\}$ in $\{1, 2, 3, 4, 5\}$ is $A' = \{4, 5\}$. The complement of A is denoted A'.

Whenever we talk of complements, we must do so in reference to a larger set. For example, the complement of $\{1, 2, 3\}$ in $\{1, 2, 3, 4, 5\}$ is $\{4, 5\}$. But the complement of $\{1, 2, 3\}$ in $\{1, 2, 3, 4, 5, 6\}$ is $\{4, 5, 6\}$.

Let A and B be subsets of the universal set $U = \{1, 2, 3, 4, 5, 6, 7\}$. If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$, then $A' = \{5, 6, 7\}$ and the relationships between the sets can be represented by the Venn diagram below. The shaded region represents A', the complement of A.



5.2 Real numbers

The set of real numbers is, roughly speaking, the set of all numbers that can be represented on a number line. It comprises the set of rational numbers (positive and negative integers, zero, and fractions) and the set of irrational numbers (non-repeating decimals like π and $\sqrt{2}$).

5.2.1 Properties of arithmetic operations on real numbers

The commutative property

An operation is said to be commutative if the order of the operands does not change the result.

For example, addition is commutative because the order in which numbers are added in a sum does not change the answer.

For example, $2 + 3 = 3 + 2$.

Similarly, $2 + 3 + 4 = 4 + 2 + 3 = 3 + 2 + 4$.

Subtraction is not commutative. This means that when subtracting numbers, order matters. Changing the order of the numbers will give a different answer.

For example, $2 - 3 \neq 3 - 2$.

Multiplication is commutative. Again, this means that when multiplying numbers, we may do so in any order.

For example, $2 \times 3 = 3 \times 2$.

Similarly, $2 \times 3 \times 4 = 4 \times 2 \times 3 = 3 \times 2 \times 4$.

Division is not commutative. Order is important when dividing. $4 \div 2 \neq 2 \div 4$.

The associative property

This property is about the grouping of numbers when doing arithmetic with three or more numbers. If changing the grouping of the operands does not change the result, the operation is associative.

For example, when adding three numbers, it does not matter how we group the numbers with parentheses:

$$(1 + 2) + 3 = 1 + (2 + 3)$$

This is because addition of numbers is associative.

Multiplication is also associative. For example, when multiplying three numbers, it does not matter how we group them:

$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$

Subtraction and division are, however, not associative. Changing the grouping of the numbers changes the result. For example,

$$(1-2)-3 = -4 \quad \text{but} \quad 1-(2-3) = 2$$

The distributive property

The distributive property of multiplication over addition is the property that when multiplying a sum by a number, we can first take the sum and multiply by the number or multiply the number by each of the addends and then take the sum of these new products.

For example,

$$2 \times (3+4) = 2 \times 7 \quad \text{and also} \quad 2 \times (3+4) = 2 \times 3 + 2 \times 4$$

5.2.2 Prime factorization

A prime number is a natural number greater than 1 whose only factors are 1 and itself.

Prime factorization is the process of breaking up a composite number into a product of prime numbers.

Every integer greater than 1 is either prime or the product of a collection of primes.

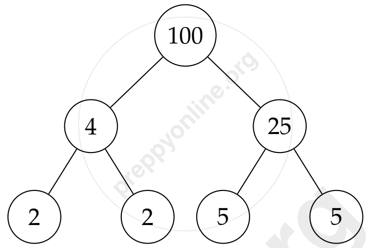
For example,

- 5 is prime.
- 15 is composite. It is the product of 3 and 5.
- 100 is composite. It is the product of the primes 2, 2, 5, and 5.
- 13 is prime.

To break an integer into a product of primes, we may first break it up into a product of any two integers and then break up those integers into a product of integers until we get a product of primes. This process can be illustrated with a factor tree.

For example, $100 = 4 \times 25$. Both 4 and 25 are not prime but can be written as products of primes: $4 = 2 \times 2$ and $25 = 5 \times 5$. Thus, $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$.

This is what the factor tree below shows.



5.2.3 Standard form

To write a number in standard form, we need to write it in the form $a \times 10^n$ where $1 \leqslant a < 10$ and n is an integer.

For example, to write 0.00459 in standard form, we need to keep moving the decimal point to the right until it is after the first non-zero number. Doing that while keeping track of the number of rimes we moved, we get

$$0.00459 = 4.59 \times 10^{-3}$$

5.3 Percentages

A percentage is a fraction of 100.

For example,

$$50\% = \frac{50}{100}$$

To express a number as a percentage, we multiply by 100%.

For example, $\frac{1}{4}$ can be expressed as a percentage as

$$\frac{1}{4} \times 100\% = 25\%$$

5.4 Simple interest

The formula for simple interest is

$$\text{simple interest} = \text{principal} \times \text{rate} \times \text{time}$$

The principal is the amount borrowed or invested.

The rate is how much the principal increases per given period. It can be stated per annum (same as per year), per month, per week, etc.

The time is how long the amount is borrowed or invested.

For example, to find the simple interest on GH¢ 2,000.00 invested at 10% simple interest per annum for a period of 9 months, we extract the relevant information.

The principal is GH¢ 2,000.00.

The rate is 10% per annum or per year.

The time is 9 months. However, because the rate is given in terms of years, we would like to convert the time also to years.

Since there are 12 months in a year, 9 months is equivalent to $\frac{9}{12}$ years.

With these three values, the simple interest may be computed as

$$\begin{aligned}\text{simple interest} &= \text{GH¢ } 2000 \times 10\% \times \frac{9}{12} \\ &= 2000 \times \frac{10}{100} \times \frac{9}{12} \\ &= \text{GH¢ } 150.00\end{aligned}$$

5.5 Ratio and proportion

5.5.1 Ratios

A ratio is a way of expressing the relationship between two numbers.

For example, if there are 4 mangoes and 6 oranges in a basket, the ratio of mangoes to oranges is 4 : 6.

This ratio may also be written as $\frac{4}{6}$. Hence, ratios and fractions can be used to represent the same information.

Just as the fraction $\frac{4}{6}$ is equivalent to $\frac{2}{3}$, the ratio 4 : 6 is equivalent to 2 : 3.

Ratios tell us how a number is divided and the relative sizes of the portions.

For example, if the ratio of red pens to blue pens in a box is 2 : 3 and there is a total of 15 pens in the box, how many are red?

The ratio 2 : 3 tells us that for every 2 red pens there are, there are 3 blue pens.

Thus, if we see 4 red pens, we should see 6 blue pens; if we see 6 red pens, there should be 9 blue pens, and so on.

The fraction of pens in the box that are red is

$$\frac{2}{2+3} = \frac{2}{5}$$

Hence, the number of red pens is

$$\frac{2}{5} \times 15 = 6$$

5.5.2 Proportions

We get a proportion when we set two ratios equal to each other.

For example, if the ratio of boys to girls in a class is 3 : 4, and there are 15 boys in the class, how many girls are there?

This information can be presented as

$$\begin{aligned}\text{boys} &: \text{girls} \\ 3 &: 4 \\ 15 &: x\end{aligned}$$

Since the three ratios above are equal, we can write

$$3 : 4 = 15 : x$$

When we do that, we get a proportion.

Since ratios can be written as fractions, we have

$$\frac{3}{4} = \frac{15}{x}$$

Cross-multiplying and solving for x gives

$$\begin{aligned}3x &= 15(4) \\ x &= \frac{15(4)}{3} \\ x &= 20\end{aligned}$$

5.5.3 Direct proportion

When the relationship between two quantities is such that as one increases, the other increases and when one decreases, the other decreases, there is direct relationship between them.

For example, if 10 cows eat 20 bags of food, how many bags of food will 20 cows eat?

20 cows will eat more food than 10 cows. In fact, they will eat twice as much food.

The information can be written as

$$\begin{aligned}10 \text{ cows} &\longrightarrow 20 \text{ bags} \\ 20 \text{ cows} &\longrightarrow x \text{ bags}\end{aligned}$$

Since there is a direct relationship between the number of cows and the number of bags, we can write these equivalent fractions:

$$\frac{10}{20} = \frac{20}{x}$$

Cross-multiplying and solving for x gives

$$\begin{aligned}10x &= 20(20) \\ x &= \frac{20(20)}{10} \\ x &= 40\end{aligned}$$

5.5.4 Indirect or inverse proportion

When the relationship between two quantities is such that as one increases, the other decreases and when one decreases, the other increases, there is indirect or inverse relationship between them.

For example, if 5 men use 20 hours to paint a house, how many hours will 10 men use to paint it?

10 men will use less time to paint the house than 5 men will. In fact, they will use half the amount of time 5 men will use.

The information can be written as

$$5 \text{ men} \rightarrow 20 \text{ hours}$$
$$10 \text{ men} \rightarrow x \text{ hours}$$

Because of the inverse relationship between the number of men and the amount of time they will need, we can write these equivalent fractions:

$$\frac{5}{10} = \frac{x}{20}$$

Notice that because of the inverse relationship between the number of men and the number of hours, we flipped the order of the quantities on the right-hand side of the proportion.

Solving for x in the proportion above gives $x = 10$.

5.6 Factorization of algebraic expressions

5.6.1 The difference of two squares

Expressions of the form $a^2 - b^2$, where one square is subtracted from another have may be factorized as

$$a^2 - b^2 = (a - b)(a + b)$$

This is an important factorization that has many applications in mathematics and should be memorized.

The factorization may be verified by expanding it:

$$(a - b)(a + b) = a^2 + ab - ab - b^2 = a^2 - b^2$$

Using the difference of two squares factorization, $4x^2 - 9y^2$ can be factorized as

$$4x^2 - 9y^2 = (2x)^2 - (3y)^2 = (2x - 3y)(2x + 3y)$$

To evaluate $65^2 - 35^2$, we may use the difference of two squares factorization to write

$$65^2 - 35^2 = (65 - 35)(65 + 35) = 30(100) = 3000$$

5.7 Mensuration

Mensuration is all about measurement. That is, the measurement of distances, areas, volumes, etc.

5.7.1 Perimeter

The perimeter of a plane figure is the sum of the lengths of its sides or the distance around its edge.

The circumference of a circle with radius r is given by the formula

$$\text{circumference} = 2\pi r$$

The perimeter of a rectangle is given by the formula

$$\text{perimeter} = 2 \times (l + w)$$

5.7.2 Area

The area of a circle with radius r is given by the formula

$$\text{area} = \pi r^2$$

The area of a rectangle with length l and width w is given by the formula

$$\text{area} = l \times w$$

The area of a square with side length a is given by the formula

$$\text{area} = a^2$$

The area of a triangle with base b and perpendicular height h is given by the formula

$$\text{area} = \frac{1}{2} \times b \times h$$

The area of a trapezium with parallel sides a and b and perpendicular height h is given by the formula

$$\text{area} = \frac{1}{2} \times (a + b) \times h$$

The area of a parallelogram with base b and perpendicular height h is given by the formula

$$\text{area} = b \times h$$

5.7.3 Volume

The volume of a cube with side length a is given by the formula

$$\text{volume} = a^3$$

The volume of a cylinder with radius r and height h is given by the formula

$$\text{volume} = \pi r^2 h$$

The volume of a sphere with radius r is given by the formula

$$\text{volume} = \frac{4}{3}\pi r^3$$

5.8 Rules of indices

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 1$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{a^n}$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

5.9 Rules of logarithms

Product rule

$$\log_b(xy) = \log_b x + \log_b y$$

For example,

$$\log_2 64 = \log_2(8 \times 8) = \log_2 8 + \log_2 8 = 3 + 3 = 6$$

Quotient rule

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

For example,

$$\log_2\left(\frac{32}{4}\right) = \log_2 32 - \log_2 4 = 5 - 2 = 3$$

Power rule

$$\log_b(x^p) = p\log_b x$$

For example,

$$\log_3 81 = \log_3 3^4 = 4\log_3 3 = 4$$

Root rule

$$\log_b \sqrt[p]{x} = \frac{\log_b x}{p}$$

For example,

$$\begin{aligned}\log_3 \sqrt{243} &= \log_3 243^{\frac{1}{2}}\\ &= \frac{1}{2}\log_3 243\\ &= \frac{1}{2}\log_3 3^5\\ &= \frac{1}{2} \times 5\\ &= \frac{5}{2}\end{aligned}$$

5.10 Surds

A surd is an irrational number expressed as a root (such as a square root or a cube root) that cannot be simplified into a whole number or exact fraction.

$\sqrt{2}$ is a surd because it cannot be simplified into a whole number or exact fraction.

$\sqrt{4}$ is not a surd because it can be simplified into the whole number 2.

$\sqrt{\frac{25}{4}}$ is not a surd because it can be simplified into the fraction $\frac{5}{2}$.

$sqrt32$ is a surd because it cannot be simplified into a whole number or exact fraction as 32 is not a perfect square.

We will restrict ourselves to square roots but the principles for other roots are similar.

5.10.1 Rules of surds

The product rule

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

For example,

$$\sqrt{2}\sqrt{2} = \sqrt{2 \times 2} = \sqrt{4} = 2$$

Also,

$$\sqrt{5}\sqrt{2} = \sqrt{10}$$

The quotient rule

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

For example,

$$\frac{\sqrt{32}}{\sqrt{2}} = \sqrt{\frac{32}{2}} = \sqrt{16} = 4$$

5.10.2 Simplification of surds

To simplify a surd, we try to write the number under the radical as a product of a perfect square and another number.

For example, since $32 = 16 \times 2$, $\sqrt{32}$ may be written as

$$\sqrt{32} = \sqrt{16 \times 2}$$

Then, applying the rules of surds, we get

$$\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$$

Similarly,

$$\sqrt{99} = \sqrt{9 \times 11} = \sqrt{9}\sqrt{11} = 3\sqrt{11}$$

5.10.3 Rationalization of the denominator

This is a technique used to eliminate surds from the denominator of a fraction. We multiply both the numerator and the denominator by a special number that the value of the fraction is not changed but the denominator becomes a rational number.

For example, $\frac{2}{\sqrt{3}}$ is the same as

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

When the denominator of the fraction is a sum or a difference involving a radical (for example, $a + \sqrt{b}$), we multiply by both the numerator and the denominator by the conjugate of the surd, which is the expression with the opposite sign between the two terms. That is, the conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$ and vice versa.

For example, to rationalize the denominator of $\frac{2}{2+\sqrt{3}}$, we multiply both the numerator and the denominator of the fraction by the conjugate of $2 + \sqrt{3}$:

$$\frac{2}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}$$

Using the fact that the denominator is now a difference of two squares (Section 5.6.1), we have

$$\begin{aligned}\frac{2\left(2-\sqrt{3}\right)}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)} &= \frac{2\left(2-\sqrt{3}\right)}{2^2 - \left(\sqrt{3}\right)^2} \\ &= \frac{4-2\sqrt{3}}{4-3} \\ &= \frac{4-2\sqrt{3}}{1} \\ &= 4-2\sqrt{3}\end{aligned}$$

5.11 Sequences and series

Sometimes a sequence of numbers follows a pattern by which we can deduce what the other numbers in the sequence are.

For example, can you find the next number in the sequence below?

$$1, 2, 4, 7, 11, \ldots$$

If we notice the pattern below, we can guess the next number as 16.

$$1 \xrightarrow{+1} 2 \xrightarrow{+2} 4 \xrightarrow{+3} 7 \xrightarrow{+4} 11 \xrightarrow{?} \ldots$$

5.11.1 Arithmetic progressions or linear sequences

An arithmetic progression (AP) or linear sequence is a sequence of numbers in which the difference between two consecutive terms is constant. This constant difference between the terms of the sequence is called the common difference.

For example, $1, 2, 3, 4, \ldots$ is an arithmetic progression because every two consecutive terms in the sequence differ by 1. In other words, given a term of the sequence, we obtain the next term by adding 1.

$2, 4, 6, 8, \ldots$ is also an arithmetic progression because the difference between every pair of consecutive terms is constant. In particular, given a term of the sequence, the next term is obtained by adding 2.

The first term and the common difference

What is the common difference of the arithmetic progression $4, 2, 0, -2, \ldots$?

Given a term of the sequence, we must add -2 to it to get the next term. Hence, the common difference is -2.

The difference between consecutive terms of an arithmetic progression is called the common difference. It is usually denoted by d.

The first term of an arithmetic progression is usually denoted by a.

Hence, in the sequence $1, 2, 3, 4, \ldots$, the first term, $a = 1$ and the common difference, $d = 1$.

In the sequence $2, 4, 6, 8, \ldots$, the first term, $a = 2$ and the common difference, $d = 2$.

The nth term of an AP

The nth term of an arithmetic progression, denoted u_n, is given by the formula

$$u_n = a + (n-1)d$$

For example, given the sequence $3, 5, 7, 9, \ldots$, since the first term, $a = 3$, and the common difference, $d = 2$, the 10th term, u_{10}, is given by

$$u_{10} = 3 + (10-1)2 = 3 + 9(2) = 3 + 18 = 21$$

Notice that we could have found the 10th term by continuing the sequence until we got to the 10th term. However, it is faster to use the formula.

5.11.2 Geometric progressions or exponential sequences

A geometric progression or exponential sequence is a sequence of numbers where successive terms are obtained by multiplying the same number, called the common ratio.

An example of a geometric progression is $1, 2, 4, 8, \ldots$. Each term in the sequence is obtained by multiplying the preceding term by the constant 2.

Another example is $81, 27, 9, 3, \ldots$. This is a geometric progression because subsequent terms are obtained by multiplying the preceding term by a constant, $\frac{1}{3}$.

The first term and the common ratio

Find the common ratio of the geometric progression $64, 32, 16, 8, \ldots$.

The common ratio is the number that we must multiply a given term by to get the next term. Given 64, the next term is 32. Since we must multiply 64 by $\frac{1}{2}$ to get 32, the common ratio is $\frac{1}{2}$.

The first term of a geometric progression is usually denoted a, while the common ratio is denoted r.

The nth term of a GP

The nth term of a geometric progression, denoted u_n, is given by the formula

$$u_n = ar^{n-1}$$

For example, since the first term of the geometric progression $1, 2, 4, 8, \ldots$ is 1 and the common ratio is 2, the 10th term of the sequence is

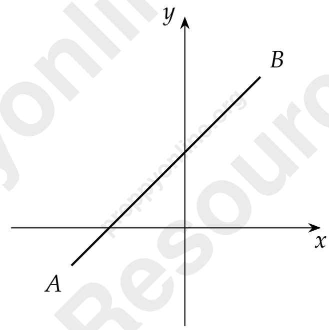
$$u_{10} = 1(2)^{10-1} = 2^9 = 512$$

5.12 Coordinate geometry

A straight line is the shortest distance between two points.

They have many applications in mathematics.

An example of a straight line is line AB below.



5.12.1 The gradient or slope of a line

Imagine climbing a mountain. The steeper the slope, the harder it is to climb; and the gentler the slope, the easier it is to climb.

The gradient or slope of a line is a measure of how steep the line is.

For example, lines A and B below have gentle slopes,

while lines C and D below have steep slopes.

To find the gradient of a line, we need to find to points (x_1, x_2) and (y_1, y_2) on the line and use the formula

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, to find the gradient of the line that passes through the points $(2, 5)$ and $(-1, 7)$, we let one pair of coordinates be (x_1, y_1) and the other one be (x_2, y_2) and use the formula to get the gradient as

$$\frac{7-5}{-1-2} = \frac{2}{-3} = -\frac{2}{3}$$

5.12.2 The equation of a line

The equation of a line may be written in different forms.

The general form of the equation of a line

The general form of the equation of a line is

$$ax + by + c = 0$$

This means that equation of every line can be written in this form. We can't have terms like x^2, x^3, y^2, and so on in the equation of a line.

The slope-intercept form of the equation of a line

The equation of a line may be written in the form

$$y = mx + c$$

This is called the slope-intercept form of the equation of a line.

When the equation of a line is written this way, we can immediately read the slope or gradient of the line and its y-intercept. The slope is m and the y-intercept is c.

The intercept form of the equation of a line

The equation of a line may be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This is called the intercept form of the equation of a line.

When the equation of a line is written this way, the x-intercept is a and the y-intercept is b.

5.12.3 How to find the equation of a line

A line is defined by two points or a point and its gradient.

Finding the equation of a line given its gradient and a point on the line

Given the gradient m of a line and a point $P(x_1, y_1)$ on the line, we can find the equation of the line as follows.

If a general point on the line has coordinates (x, y) then we have

$$\frac{y-y_1}{x-x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1)$$

From this, we can get the equation of the line.

For example, if a line has gradient 2 and passes through the point $(3, 4)$, then we can find its equation by writing

$$\begin{aligned} \frac{y-4}{x-3} &= 2 \\ y - 4 &= 2(x-3) \\ y - 4 &= 2x - 6 \\ y &= 2x - 2 \end{aligned}$$

Finding the equation of a line given two points on the line

Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a line, we can find the equation of the line by first using the two points to find the gradient. After that, we use one of the points and the gradient we found to find the equation of the line.

The gradient of the line is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Using the point $P(x_1, y_1)$ and the gradient, we can find the equation of the line by writing

$$\begin{aligned} \frac{y-y_1}{x-x_1} &= \frac{y_2-y_1}{x_2-x_1} \\ y - y_1 &= \frac{y_2-y_1}{x_2-x_1}(x - x_1) \end{aligned}$$

For example, to find the equation of the line that passes through $(1, 2)$ and $3, 5$, we first find the gradient as

$$\frac{5-2}{3-1} = \frac{3}{2}$$

Using this gradient and the point $(1, 2)$, we can find the equation of the line as

$$\begin{aligned}\frac{y-2}{x-1} &= \frac{3}{2}\\ y-2 &= \frac{3}{2}(x-1)\\ y-2 &= \frac{3}{2}x - \frac{3}{2}\\ y &= \frac{3}{2}x - \frac{3}{2} + 2\\ y &= \frac{3}{2}x + \frac{1}{2}\end{aligned}$$

5.12.4 The distance between two points

Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the distance between them is given by the formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This distance is the same as the length of the line segment between the two points.

For example, to find the distance of the between $C(2, 4)$ and $D(6, 1)$, we use the formula to get

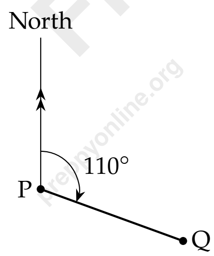
$$\begin{aligned}\sqrt{(2-6)^2 + (4-1)^2} &= \sqrt{(-4)^2 + 3^2}\\ &= \sqrt{16+9}\\ &= \sqrt{25}\\ &= 5 \text{ units}\end{aligned}$$

5.13 Bearings

Bearings are measured clockwise from due north.

For example, if the bearing or Q from P is 110°, that means that if we stand at P and start measuring clockwise from the North, Q is going to be on the line that is 110° from the North.

The figure below shows the locations of P and Q.



By convention, bearings are written with three digits. Hence, a bearing of 80° is written as 080°.

5.13.1 Back bearings

Given the bearing of Q from P, we can find the bearing of P from Q, also known as the back bearing, as follows:

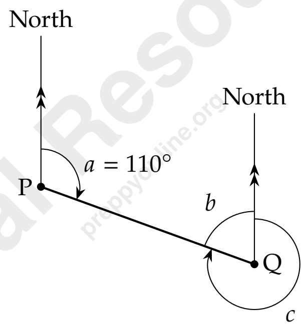
If the given bearing, x, is such that $x < 180°$, add 180° to it.

If the given bearing, x, is such that $x \geqslant 180°$, subtract 180° from it.

For example, if the bearing of Q from P is 110°, then, using the formula described above, the bearing of P from Q is $110° + 180° = 290°$.

But we could also find the back bearing by drawing a diagram and using the relationships between the angles to figure it out.

The information given can be represented with the diagram below.



The diagram shows the bearing of Q from P as angle a, while the bearing of P from Q is represented by angle c.

Because angles a and b are co-interior angles, they add up to 180°. So,

$$\begin{aligned}a + b &= 180°\\ 110° + b &= 180°\\ b &= 180° - 110°\\ b &= 70°\end{aligned}$$

Also, because angles b and c are angles at a point, we have

$$\begin{aligned}b + c &= 360°\\ 70° + c &= 360°\\ c &= 360° - 70°\\ c &= 290°\end{aligned}$$

Hence, the bearing of P from Q is 290°.

5.14 Vectors

Vectors are usually written in component form. For example, $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. The entries in the vector are called components.

In $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, a is called the x component while b is called the y component.

5.14.1 Rules of vector algebra

Vectors can be added, subtracted, and multiplied by numbers.

Addition of vectors

To add two vectors, we add the corresponding components.

For example, if $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2+1 \\ 3+2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \end{aligned}$$

Subtraction of vectors

Subtraction of vectors is also done component by component.

For example, if $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{aligned} \mathbf{v} - \mathbf{w} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2-1 \\ 3-2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \end{aligned}$$

Scalar multiplication

Vectors can be multiplied by numbers. The number by which a vector is multiplied is called a scalar.

If the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is multiplied by the scalar k, we get

$$k\mathbf{v} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

That is, each component of the vector is multiplied by the number.

For example,

$$2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2(-3) \\ 2(4) \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

5.14.2 The length of a vector

Given a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, the length of $\mathbf{v}$, denoted $|\mathbf{v}|$, is given by the formula

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

For example, the length of the vector $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is

$$|\mathbf{a}| = \left|\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right| = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13} \text{ units}$$

5.15 Trigonometry

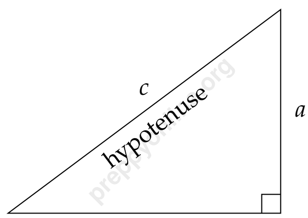
Trigonometry is about the measurement of the sides and angles of triangles.

Right-angled triangles are of special interest here because, the the basic trigonometric ratios (sin, cos, tan) are easy to define in terms of the sides of such a triangle.

5.15.1 Pythagoras' theorem

Given a right angled triangle like the one below, the longest side is called they hypotenuse.

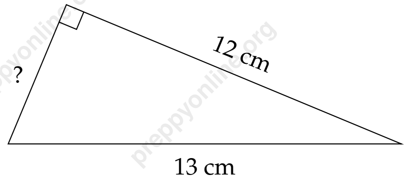
The hypotenuse is always opposite the 90° angle in the triangle.



Pythagoras' theorem states that, if the length of the hypotenuse is c and the lengths of the other sides are a and b, then the relationship between the lengths is

$$c^2 = a^2 + b^2$$

For example, since the triangle below is a right-angled triangle with hypotenuse of length 13 cm, we can use Pythagoras' theorem to find the third side as follows.



Let the length of the third side be x. Then, by Pythagoras' theorem,

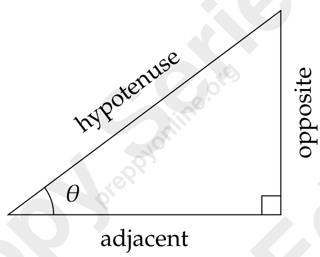
$$\begin{aligned} x^2 + 5^2 &= 13^2 \\ x^2 + 25 &= 169 \\ x^2 &= 169 - 25 \\ x^2 &= 144 \\ x &= 12 \end{aligned}$$

Thus, the length of the third side is 12 cm.

5.15.2 SOH-CAH-TOA

When one of the angles in a right-angled triangle is of interest, we use that as a reference angle to name the sides of the triangle.

The side opposite the reference angle is called the opposite side, while the side next to the reference angle (that is not the hypotenuse) is called the adjacent side. The longest side, the hypotenuse, remains the hypotenuse.



With these labels, the basic trigonometric ratios as follows.

The sine of θ is given by

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

The cosine of θ is given by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

The tangent of θ is given by

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

These are shortened by the mnemonic SOH-CAH-TOA.

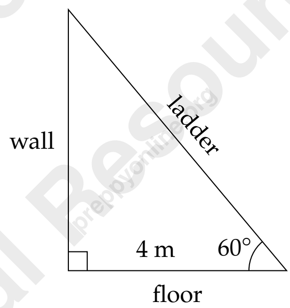
SOH means $S = \frac{O}{H}$, where S stands for sin, O stands for "opposite," and H stands for "hypotenuse."

CAH means $C = \frac{A}{H}$, where C stands for cos, A stands for "adjacent," and H stands for "hypotenuse."

TOA means $T = \frac{O}{A}$, where T stands for tan, O stands for "opposite," and A stands for "adjacent."

A ladder leans against a vertical wall so that it makes an angle of 60° with the ground. If the foot of the ladder is 4 metres from the wall, find the length of the ladder.

The sketch below shows the relationship between the ladder, the wall, and the floor.



Let the length of the ladder be x. Then, from the sketch,

$$\cos 60^\circ = \frac{4\text{ m}}{x}$$

Thus,

$$x = \frac{4\text{ m}}{\cos 60^\circ}$$

If we know the value of $\cos 60^\circ$, we can use it to calculate the length of the ladder.

Using a calculator, we find $\cos 60^\circ$ to be $\frac{1}{2}$. Hence, the length of the ladder is

$$\frac{4\text{ m}}{\frac{1}{2}} = 2(4\text{ m}) = 8\text{ m}$$

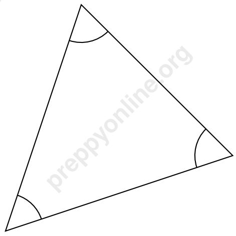
5.16 Plane geometry

5.16.1 Types of angles

Acute angles

An acute angle is an angle that measures greater than 0° and less than 90°.

Acute angles are known for their sharpness or pointedness.

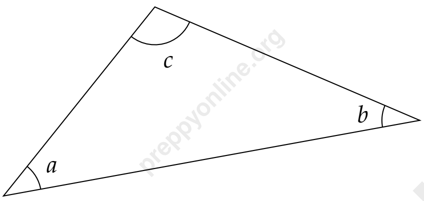


All the angles in the triangle above are examples of acute angles.

Obtuse angles

An obtuse angle is an angle that measures greater than 90° and less than 180°.

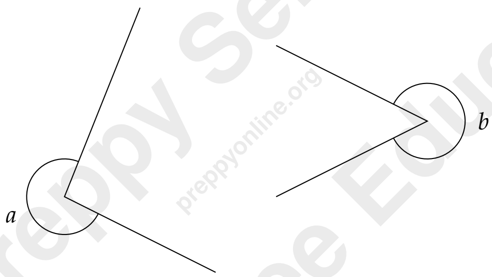
Angle c below is an obtuse angle, while angles a and b are acute angles.



Reflex angles

A reflex angle is an angle that measures greater than 180° and less than 360°.

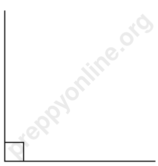
Examples of reflex angles are angles a and b below.



Right angles

A right angle is an angle that measures exactly 90°.

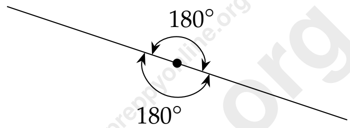
It is the angle at the corner of a square or a perfect "L" shape.



Straight angles

A straight angle is an angle that measures 180°.

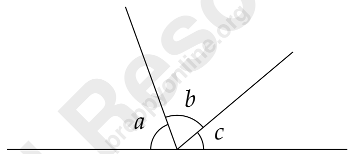
They are angles about a point on either side of a straight line.



Angles on a straight line

Since a straight angle measures exactly 180°, when it is divided up, the measures of the parts sum up to 180°.

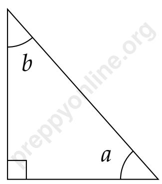
For example, angles a, b, and c add up to 180° as they are angles on a straight line.



Complementary angles

Two angles are called complementary angles if they add up to 90°.

An example of complementary angles is the two acute angles in a right-angled triangle.



Since the angles in a triangle add up to 180°,

$$a + b + 90° = 180°$$

Hence,

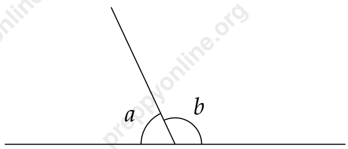
$$a + b = 180° - 90° = 90°$$

Therefore, a and b are complementary angles.

Supplementary angles

Supplementary angles are a pair of angles whose measures add up to 180°.

An example of supplementary angles are two angles on a straight line like angles a and b below.

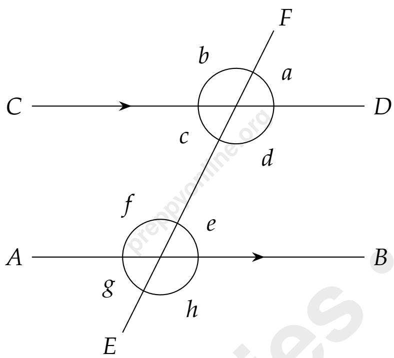


5.16.2 Angles in parallel lines

In the diagram below, lines AB and CD are parallel, as indicated by the arrows on them.

Line EF cuts through lines AB and CD. It is called a transversal.

A transversal is a line that crosses two or more lines in a plane.



Whenever a transversal crosses a pair of parallel lines, it creates 8 angles with special relationships between them.

We shall discuss the relationships between the 8 angles a, b, c, d, e, f, g, h below.

Alternate angles

Angles on opposite sides of the transversal and between the parallel lines are equal.

These are called alternate angles.

In the diagram above, c and e are alternate angles. So are d and f.

Corresponding angles

Angles in the same relative position at each intersection are equal.

For example, the angle at the top-left of the intersection at the top, b, is equal to the angle at the top-left of the intersection at the bottom, f.

Co-interior angles

Angles on the same side of the transversal and between the parallel lines add up to 180°.

These are called co-interior or allied angles. They are also called consecutive interior angles.

Examples are angles d and e in the diagram.

c and f are also examples of co-interior angles.

Vertically opposite angles

Angles opposite each other at the intersections are equal.

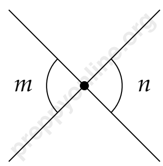
These are called vertically opposite angles.

Examples include a and c in the diagram above.

Angles b and d are also vertically opposite and therefore equal.

Vertically opposite angles are called "vertically" opposite because they are opposite each other across their common vertex—the point where two lines meet or cross—and not because they are oriented up and down. Thus, here, "vertical" is the adjectival form of "vertex."

In the diagram below, angles m and n are vertically opposite angles even though they are not oriented up and down.



5.17 Statistics

5.17.1 Types of data

Quantitative data

Quantitative data are values that can be measured numerically. Examples include

- length (as it can be measured in numbers like metres, centimetres, inches, etc.)
- weight (as it can be measured in numbers like kilograms, pounds, etc.)
- time (as it can be measured in numbers like hours, minutes, seconds, etc.)
- speed (as it can be measured in numbers like metres per second, kilometres per hour, etc.)
- temperature (as it can be measured in numbers like degrees Celsius, degrees Fahrenheit, etc.)

Qualitative data

Qualitative data are values that describe qualities or attributes rather than numerical measurements. Such data can typically be arranged into categories or groups. Examples include

- colour (red, blue, green, etc.)
- marital status (single or married)
- nationality (Ghanaian, Nigerian, Japanese, etc.)

5.17.2 Measures of central tendency

These give a single number by which the data can be represented. Examples include the mean, the median, and the mode.

The mean

The mean of a list of numbers is given by the sum of the numbers divided by how many numbers there are.

For example, the mean of the numbers $1, 1, 2, 2, 2, 2, 3, 3, 4$ is

$$\frac{1+1+2+2+2+2+3+3+4}{9} = \frac{20}{9}$$

The median

The median is the middle number when data is arranged in ascending or descending order.

For example, in the data $1, 1, 2, 2, 2, 2, 3, 3, 4$, the median is 2 because 2 is the middle number when the numbers are arranged in ascending or descending order.

The mode

Given some data, such as a list of numbers, the mode is the data that occurs most frequently.

For example, in the data $1, 1, 2, 2, 2, 2, 3, 3, 4$,

1 appears 2 times.

2 appears 4 times.

3 appears 2 times.

4 appears once.

Hence, the mode is 2 as 2 is the data that appears most frequently.

5.17.3 Measures of dispersion

These give an idea of spread out the data is.

Examples include the range and the standard deviation.

Range

Given a list of numbers, the range is the difference between the largest and the smallest.

For example, in the list $1, 1, 2, 2, 2, 2, 3, 3, 4$, the range is

$$4 - 1 = 3$$

since the largest number in the list is 4 and the smallest number is 1.

5.18 Loci

A locus is a path traced by a moving point.

There are many rules by which a point could move, and each of them gives a locus.

We shall mention some important examples below.

5.18.1 Examples of loci

The circle

If a point moves so that it is always the same distance from a given point, the moving point will trace a circle with the given point as its centre.

The perpendicular bisector

The set of points that are equidistant from two given points will trace the perpendicular bisector of the line joining the two given points.

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